بِسْمِ اللهِ الرَّحْمٰنِ الرَّحِيْمِ

“In the name of Allah ,the Most Beneficent ,the Most Merciful.”

شروع کوم په نامه د اللّٰه(ج) چې رحمٰن او رحيم دی.

Anas bin Malik (رضي الله عنه) narrated that Allah’s Messenger(صلى الله عليه وسلم) said: “Indeed I hear the crying of a small boy while I am in Salat so I shorten it for fear that this mother may be tormented,” **(Sahih)**

[H 376 Jami At-Tirmidhii Arabic-English, Vol 1, Compiled by Imam Hafiz Abu Eisa At Termidhi, Translated by: Abu Khaliyl (USA)]

ګرانو او قدرمنو ورونو اوخويندو! السلام عليکم ورحمت الله وبرکاته!

له الله(ج) څخه هیله کوم چې تاسې ټول روغ او جوړ وئ. دغه لاندې د رياضي درسونه به ان شاءالله ځينې پلرونو او په خاصه توګه، د دوی اولادونو ته ډېر ګټور وي.

نوټ: دغه لړی به ان شاءالله، له لومړي ټولګي څخه تر پوهنتون پوري ان شاءالله، دوام ولري.
د يادونې وړ ده، چې زما مسلک رياضي نده، او ما خپلې زده کړې، د دکتورا تر کچې د الکترانکس په برخه کې، په انګلستان کې سرته رسولې. همدارنګه، ما ۶ کاله د کابل پوهنتون، د انجينرۍ په پوهنځی، او ۲۷ کاله د انګلستان په يوه پوهنتون کې معلمي کړېده. يعنې هغه چا چې په الکترانکس انجينرنګ کې زده کړې کړې وي، نو هغه د رياضي په قدر پوهيږي. زه به له رياضي درسونو وروسته، د الکترانکس په برخه کې به ان شاءالله درسونه شروع کړم.

Dear Brothers and Sisters! السلام عليکم و رحمت الله و برکاته

I kindly request each one of you to remember me, my parents, my children, my family and the entire Muslims of the world in your daily Prayers.

ګرانو او قدرمنو ورونو اوخويندو!
السلام عليکم ورحمت الله وبرکاته!

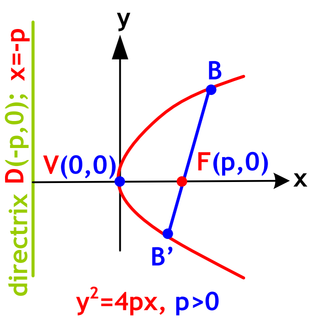
له الله(ج) څخه هیله کوم چې تاسې ټول روغ او جوړ وئ، او الله(ج) د وکړي، چې ماته ، زما مور او پلار ته، زما اولادونو ته ، زما ټولې کورنۍ ته او د نړۍ ټولو مسلمانانو ته ، د زړه له کومې دعا وکړئ. والسلام

زما اړيکه : **abdullahwardak53@gmail.com**

Chapter-30: Parabolas and Hyperbolas

1. Parabolas:

The locus of a point **P** which moves in a plane so that its distance from a fixed line of the plane and its distance from a fixed point of the plane, not on the line, are equal is called a parabola. The fixed point **F** is called the **focus**, and the fixed **line d** is called the directrix of the parabola. The line **FD** through the focus and perpendicular to the directrix is called the **axis of the parabola**. The axis intersects the parabola in the point **V**, the midpoint of **FD**, called the **vertex**. The line segment joining any two distinct points of the parabola is called a **chord**. A chord as **BB'** which passes through the focus is called a **focal chord**, while **FB** and **FB'** are called the focal radii of **B** and **B'**, respectively as shown **below**.



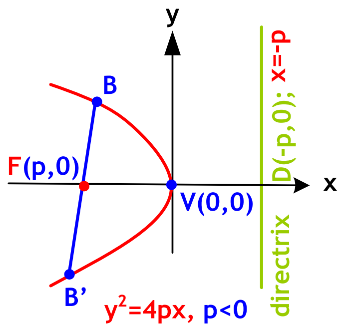
F = Focus

V = Vertex = $\frac{FD}{2}$

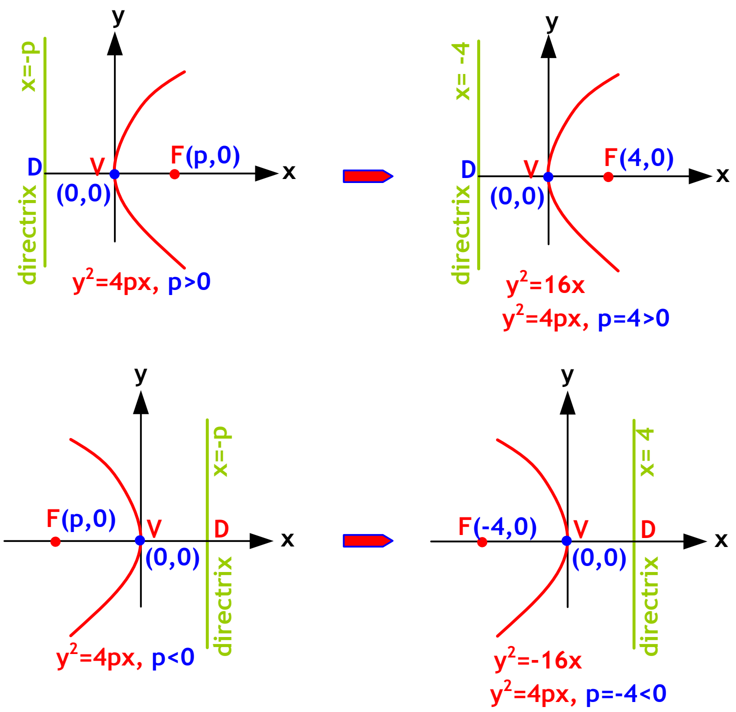
BB' = Focal Chord

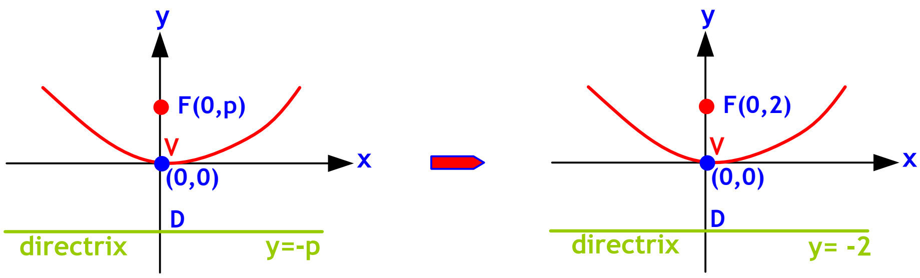
FB and FB' = The Focal Radii of B and B'

FD = The axis of the Parabola



When the vertex is at the **origin** and the axis coincides with the **x axis**, the equation of the parabola is $y^2 = 4px$. The Focus is **F(p,0).** The directrix is **D(-p,0)**. If **p>0**, the parabola opens to the right. If **p<0**, the parabola opens to the left.

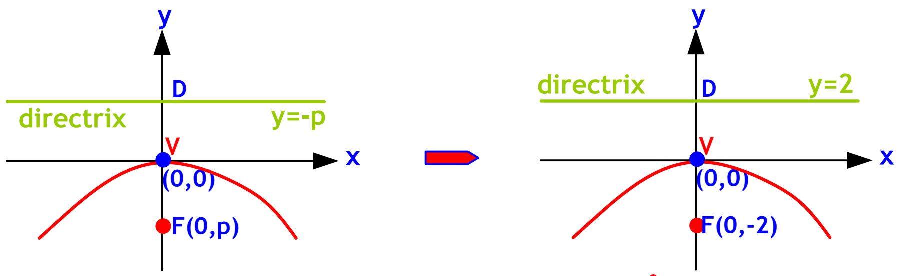




$x^2=4py$, $p>0$

$x^2=8y$

$x^2=4py$, $p=2>0$



$x^2=4py$, $p<0$

$x^2=-8y$

$x^2=4py$, $p= -2<0$

Note: The **equation** of a **parabola** when its **vertex** is at the point **(h,k):**

$$(x-h)^2 = 4p(y-k) \quad ...(1)$$ **along y-axis(i.e. opens upward/downward).**

$$(y-k)^2 = 4p(x-h) \quad ...(2)$$ **along x-axis(i.e. opens right/left).**

Example: Find the vertex, focus and the equation of the axis and directrix of the parabola $x^2-8y-4x-20=0$

Ans: $x^2-4x-8y-20=0$

$$x^2-4x+4-4-8y-20=0$$

$$(x-2)^2-8y-24=0$$

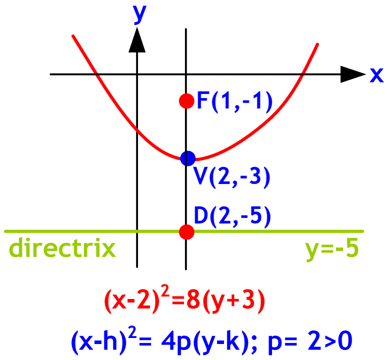
$$(x-2)^2-8(y+3)=0$$

$$(x-2)^2=8(y+3) \quad ...(1)$$

$$(x-h)^2=4p(y-k) \quad ...(2)$$

Comparing **Eq.1** and **Eq.2**:

$h= 2$, $k= -3$, $4p = 8 \Rightarrow p = 2>0$



Example: Find the vertex, focus and the equation of the axis and directrix of the parabola $x^2+8y-4x-20=0$

Ans: $x^2-4x+8y-20=0$

$x^2-4x+4-4+8y-20=0$

$(x-2)^2+8y-24=0$

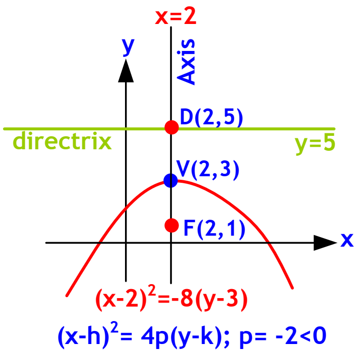
$(x-2)^2+8(y-3)=0$

$(x-2)^2=-8(y-3)$ ...(1)

$(x-h)^2=4p(y-k)$ ...(2)

Comparing Eq.1 and Eq.2:

h= 2, k= 3, 4p = -8 $\Rightarrow$ p = -2<0



Example: Find the vertex, focus and the equation of the axis and directrix of the parabola $y^2+8x-6y+41=0$

Ans: $y^2-6y+8x+41=0$

$y^2-6y+9-9+8x+41=0$

$$(y-3)^2+8x+32=0$$

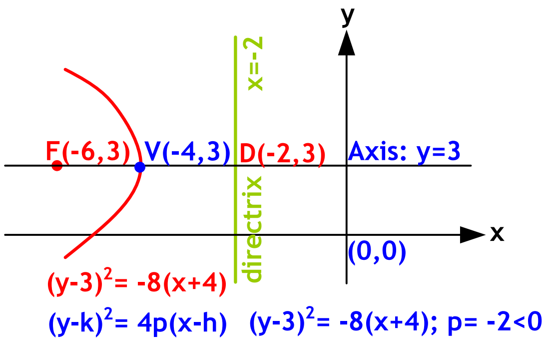
$$(y-3)^2+8(x+4)=0$$

$$(y-3)^2=-8(x+4) \qquad \text{...(1)}$$

$$(y-k)^2=4p(x-h) \qquad \text{...(2)}$$

Comparing **Eq.1** and **Eq.2**:

k = 3, h = -4, 4p = -8 $\Rightarrow$ p = -2<0



Example: Find the vertex, focus and the equation of the axis and the directrix of the parabola $y^2-8x-6y-23=0$

Ans: $y^2-6y-8x-23=0$

$$y^2-6y+9-9-8x-23=0$$

$$(y-3)^2-8x-32=0$$

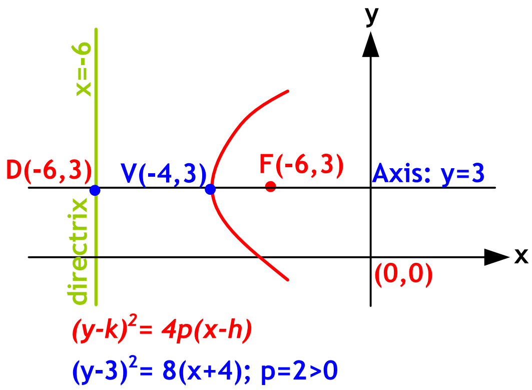
$$(y-3)^2-8(x+4)=0$$

$$(y-3)^2=8(x+4) \qquad \text{...(1)}$$

$$(y-k)^2=4p(x-h) \qquad \text{...(2)}$$

Comparing **Eq.1** and **Eq.2**:

k = 3, h = -4, 4p = 8 $\Rightarrow$ p = 2>0



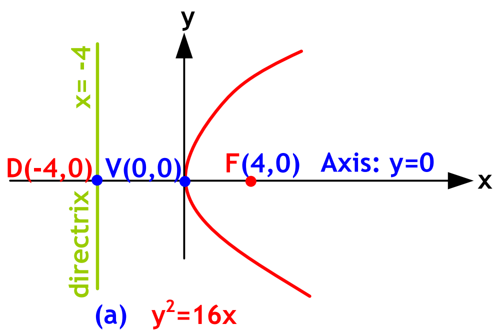
Example: For each of the following parabolas, sketch the curve, find coordinates of the vertex and focus, and find the equation of the axis and directrix.

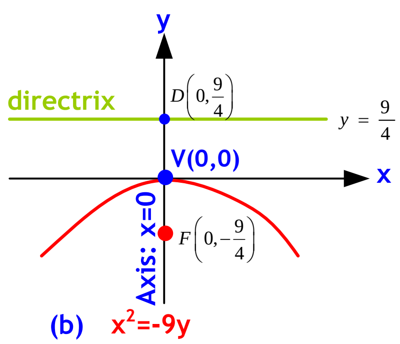
(a) $y^2 = 16x$ **(b)** $x^2 = -9y$

(c) $x^2 - 2x - 12y + 25 = 0$ **(d)** $y^2 + 4y + 20x + 4 = 0$

Ans:

(a) and (b)





(c) $x^2 - 2x - 12y + 25 = 0$

$x^2 - 2x + 1 - 1 - 12y + 25 = 0$

$(x-1)^2 = 12(y-2)$ **...(1)**

$(x-h)^2 = 4p(y-k)$ **...(2)**

Comparing **Eq.1** and **Eq.2**:

$h = 1$, $k = 2$, $\Rightarrow$ Vertex at **(1,2) and** 4p = 12 $\Rightarrow$ p = 3>0

(d) $y^2+4y+20x+4=0$

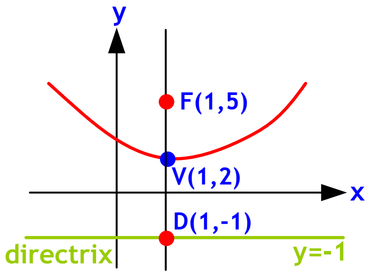
$y^2+4y+4-4+20x+4=0$

$(y+2)^2=-20x$ ...(1)

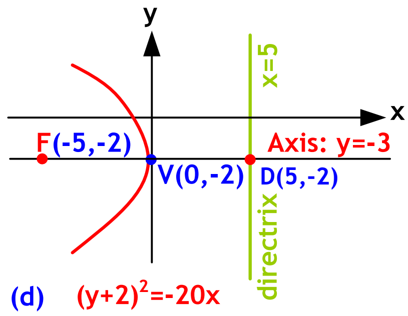
$(y-k)^2=4p(x-h)$ ...(2)

Comparing **Eq.1** and **Eq.2**:

k = -2, h = 0, $\Rightarrow$Vertex at **(0,-2)** and 4p = -20 $\Rightarrow$p = -5<0



(c) $(x-1)^2=12(y-2)$



(d) $(y+2)^2=-20x$

Example: Find the equation of the parabola, given

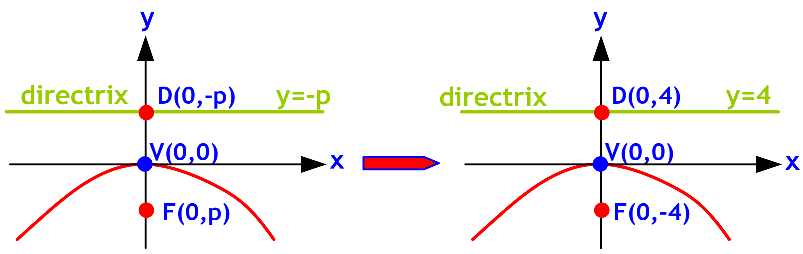
(a) V(0,0); F(0,-4) **(b)** V(0,0); directrix: x=-5

(c) V(1,4); F(-2,4) **(d)** F(2,3); directrix: y=-1

(e) V(0,0); axis: y=0; passing through (4,5)

Ans:

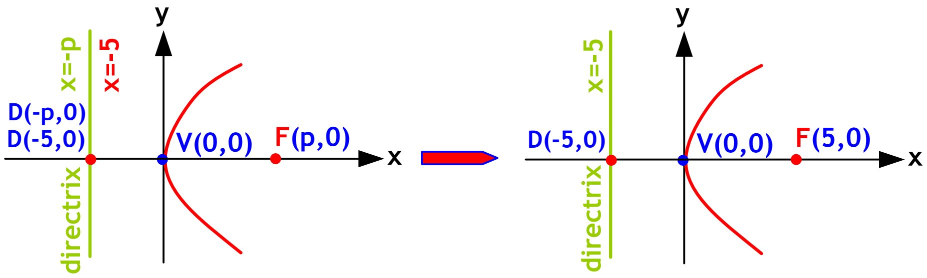
(a) **Note:** Since the directed distance **p=VF=-4,** the parabola opens downward, and the equation is: $x^2=4py=-16y$.



(a) $x^2=-16y$
$x^2=4py$

(a) $x^2=-16y$
$x^2=4py$; $p=-4<0$

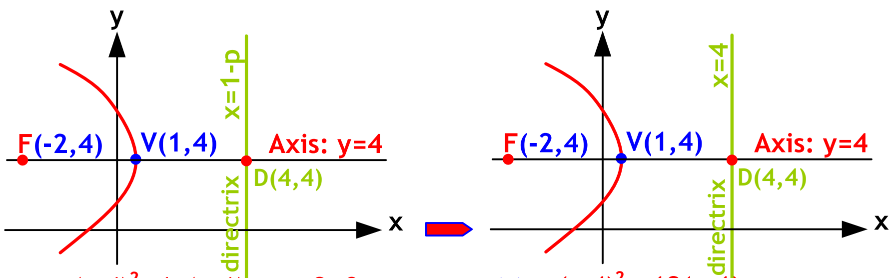
(b) **Note:** Here the directrix lies to the left of the vertex, so the parabola opens to the right. The directed distance **$p=DV=5$** and the equation is: **$y^2=4px=20x$.**



(b) $y^2=20x$
$y^2=4px;\ p>0$

(b) $y^2=20x$
$y^2=4px;\ p=5>0$

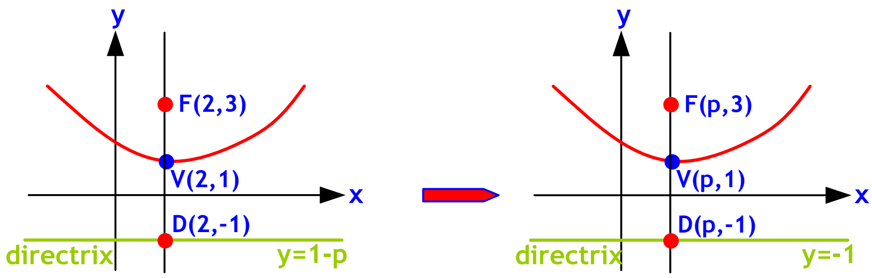
(c) **Note:** Here the focus lies to the left of the vertex, and the parabola opens to the left. The directed distance **$p=VF=-3$** and the equation is: **$(y-4)^2=4p(x-1)=-12(x-1)$.**



(c) $(y-4)^2=4p(x-1);\ p=-3<0$

(c) $(y-4)^2=-12(x-1)$

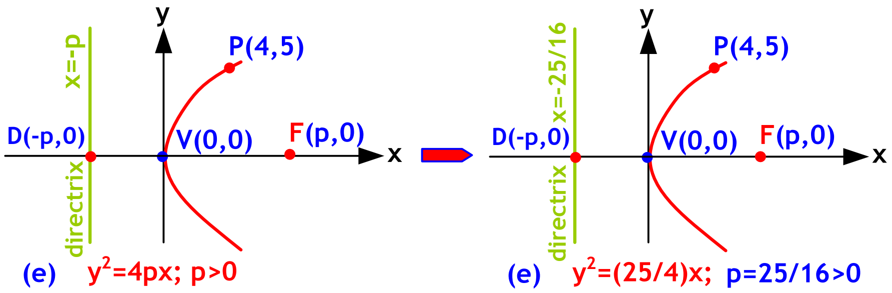
(d) **Note:** Here the focus lies above the directrix and the parabola opens upward. The axis of the parabola meets the directrix in **D(2,-1)** and the vertex is at the midpoint **V(2,1)** of **FD**. Then **$p=VF=2$** and the equation is: **$(x-2)^2=4p(y-1)=8(y-1)$.**



(d) $(x-2)^2=4p(y-1);\ p=2>0$

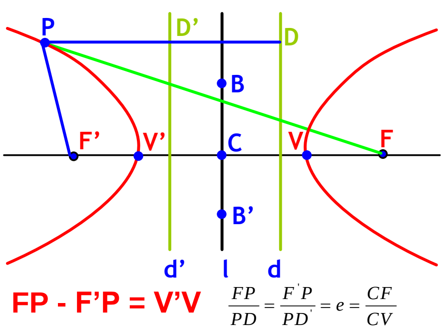
(d) $(x-2)^2=8(y-1)$

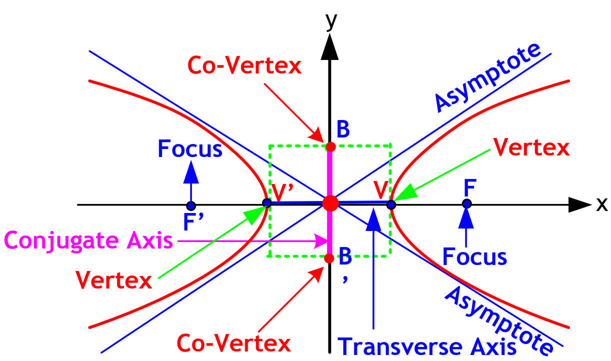
(e) **Note:** The equation of this parabola is of the form $y^2=4px$. If **(4,5)** is a point on the parabola, then $(5)^2=4p(4)$, → $16p=25$, → $p=25/16$, and the equation is: $y^2=4px=4(25/16)x=(25/4)x$.



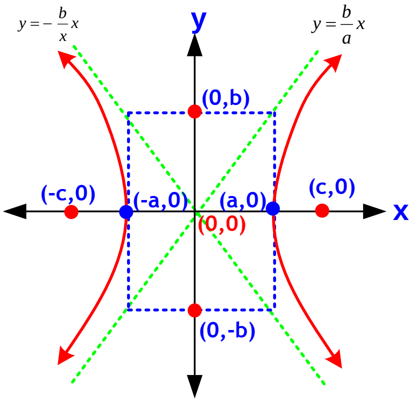
2. The Hyperbolas

The locus of a point **P** which moves in a plane so that the absolute value of the difference of its distance from two fixed points is constant is called a hyperbola. (Note that the locus consists of two distinct branches, each of indefinite length.) The fixed point **F** and F' are called the **foci**, and their midpoint C is called the centre of the hyperbola. The line **FF'** joining the foci intersects the hyperbola in the points **V** and **V'**, called the vertices.
The segment **V'V** intercepted on the line **FF'** by the hyperbola is called the its transverse axis. The line l through **C** and perpendicular to **FF'** does not intersect the curve, but it will be found convenient to define a certain segment **B'B** on **l**, having **C** as midpoint, as the conjugate axis
The line segment whose extremities are any two points (both on the same branch or one on each branch) on the hyperbola is called a **chord**. A chord which passes through a focus is called a **focal chord**.
The hyperbola may also be defined as the locus of a point which moves so that the ratio of its distance from a fixed point and its distance from a fixed line is equal to **e>1**. The fixed point is a focus **F** or **F'** and the fixed line d or **d'** is called a directrix. The ratio **e** is called the eccentricity of the hyperbola.





Note: For further understanding the following cases are presented. The reader is kindly requested to pay lots of attention to these examples.



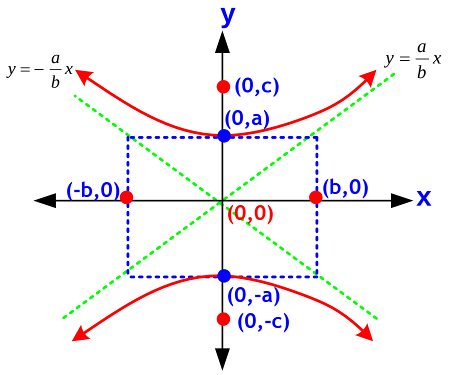
Note: When the centre is at the **origin** and the transverse axis lies along the **x-axis**, the equation of the hyperbola is:

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

Then the vertices are at **V(a,0)** and **V'(-a,0)** and the length of the transverse axis is **VV' = 2a**. The extremities of the conjugate axis are **B'(0,-b)** and **B(0,b)**, and its length is **B'B=2b.**

The foci are on the transverse axis at **F'(-c,0)** and **F(c,0)** where

$c=\sqrt{a^2+b^2}$ **and** the **eccentricity** is: $e=\frac{c}{a}=\frac{\sqrt{a^2+b^2}}{a}$



Note: When the centre is at the **origin** and the transverse axis lies along the **y-axis**, the equation of the hyperbola is:

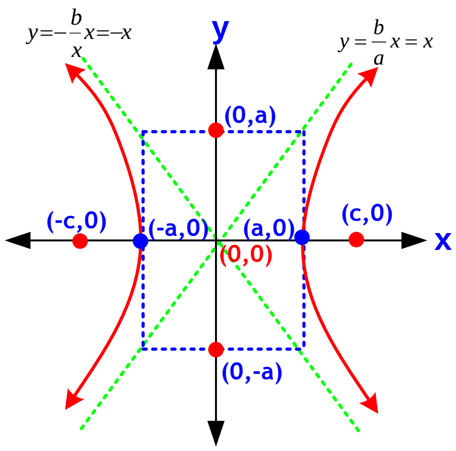
$$\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$$

Then the vertices are at **V(0,a)** and **V'(0,-a)** and the length of the transverse axis is **VV' = 2a**. The extremities of the conjugate axis are **B'(-b,0)** and B(b,0), and its length is **B'B = 2b.**

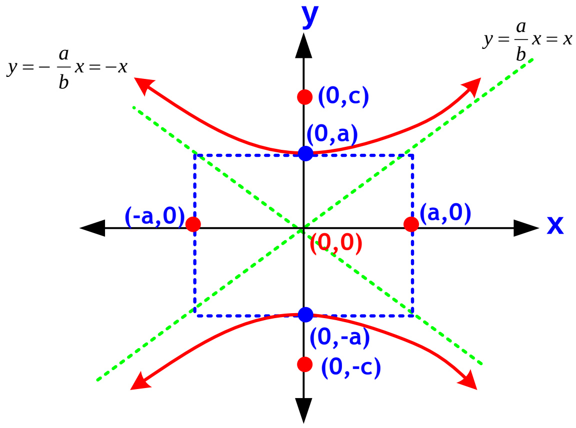
The foci are on the transverse axis at **F'(0,-c)** and **F(0,c)** where

$c=\sqrt{a^2+b^2}$ **and** the **eccentricity** is: $e=\frac{c}{a}=\frac{\sqrt{a^2+b^2}}{a}$

Note: The hyperbola whose **transverse** and **conjugate** axes are of equal length ***2a***, are called **equilateral hyperbolas**. They are also called the **rectangular hyperbolas**.

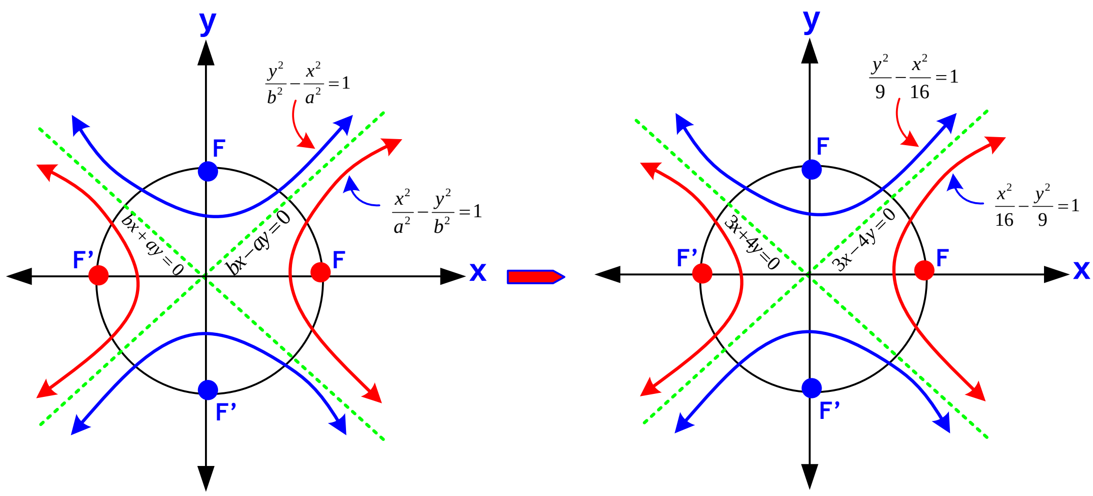


$$\frac{x^2}{a^2}-\frac{y^2}{a^2}=1$$



$$\frac{y^2}{a^2}-\frac{x^2}{a^2}=1$$

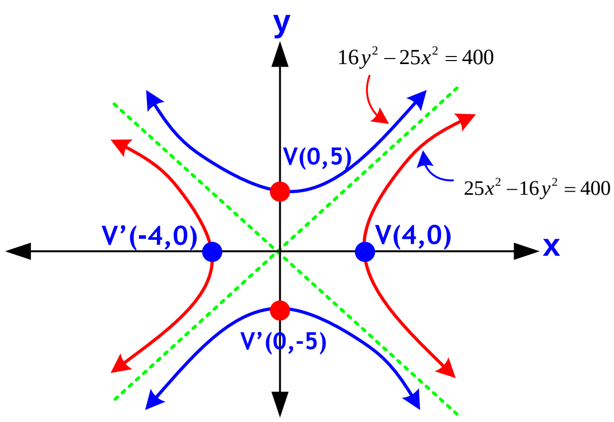
Note: Two hyperbolas such that the **transverse** of each is the **conjugate** axis of the other are called **conjugate hyperbolas**, each being the conjugate of the other.



Example: Write the equation of the conjugate of the hyperbola.

$25x^2-16y^2=400$

Note: The equation of the **conjugate hyperbola** is: $16y^2-25x^2=400$. The common asymptotes have equations $y=\pm\frac{5}{4}x$. The vertices of $25x^2-16y^2=400$ are at $(\pm 4,0)$. The vertices of $16y^2-25x^2=400$ are at $(0,\pm 5)$. The curve is shown below.



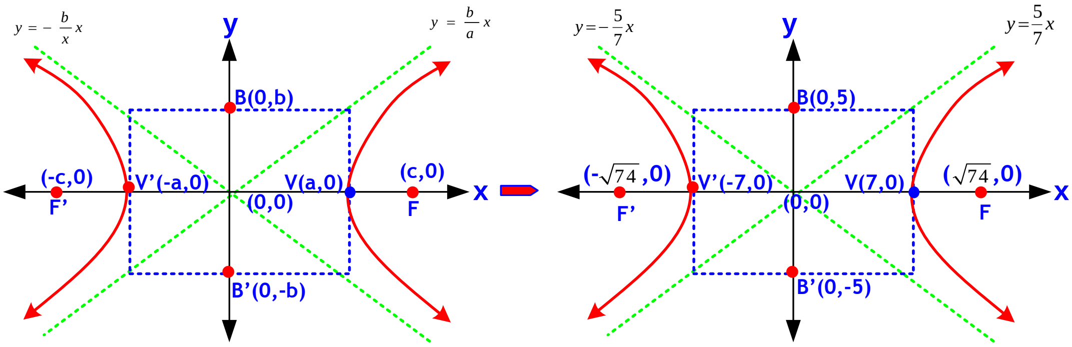
Example: Find the vertices and foci of the hyperbola with equation: $\frac{x^2}{49}-\frac{y^2}{25}=1$

Ans:

$$\frac{x^2}{49}-\frac{y^2}{25}=1 \qquad ...(1)$$

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \qquad ...(2)$$

By comparison: $a=\sqrt{49}=7,\ b=\sqrt{25}=5,\ c=\sqrt{a^2+b^2}=\sqrt{49+25}=\sqrt{74}$



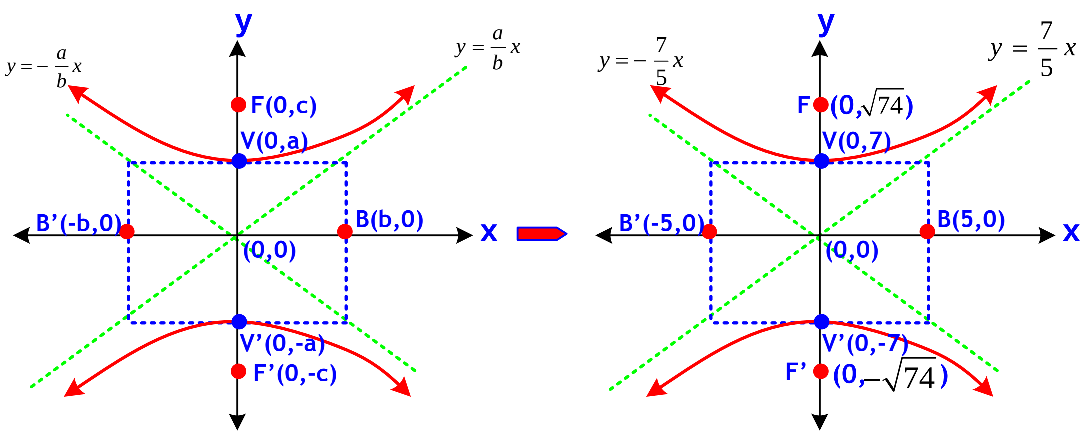
Example: Find the vertices and foci of the hyperbola with equation: $\frac{x^2}{25}-\frac{y^2}{49}=1$

Ans:

$$\frac{x^2}{25}-\frac{y^2}{49}=1 \qquad ...(1)$$

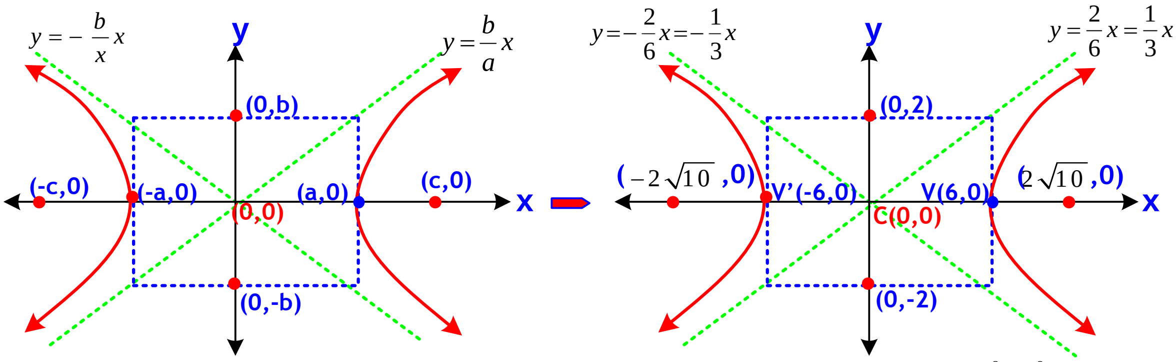
$$\frac{x^2}{b^2}-\frac{y^2}{a^2}=1 \qquad ...(2)$$

By comparison: $b=\sqrt{25}=5,\ a=\sqrt{49}=7,\ c=\sqrt{a^2+b^2}=\sqrt{49+25}=\sqrt{74}$



Example: Find the equation of a hyperbola that has vertices at $(\pm 6,0)$ and foci at $\left(\pm 2\sqrt{10},0\right)$.

Ans: Since vertices and foci are on the **x-axis**, then $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$

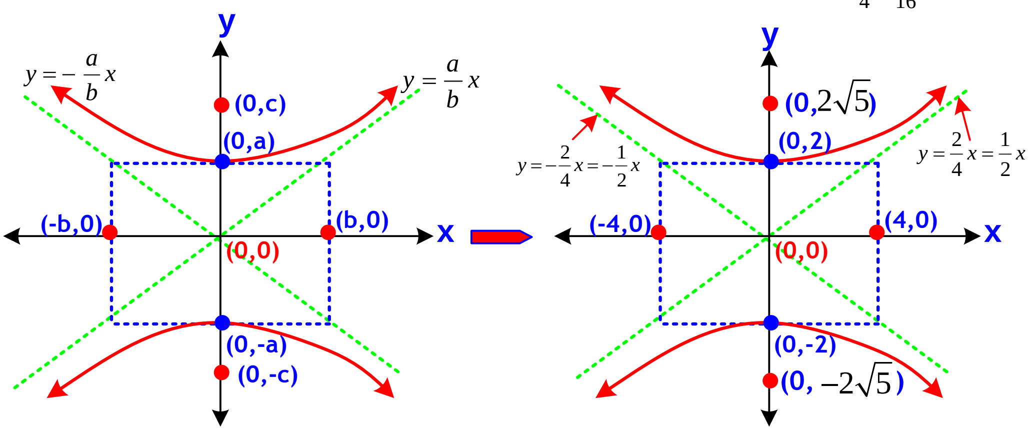


2a=V'V=12→ a=6, $c=CF=2\sqrt{10}\Rightarrow b=\sqrt{c^2-a^2}=\sqrt{40-36}=\sqrt{4}=2$ Ans: $\Rightarrow \dfrac{x^2}{36}-\dfrac{y^2}{4}=1$

Example: Find the equation of a hyperbola that has vertices at $(\pm 2,0)$ and foci at $\left(\pm 2\sqrt{5},0\right)$:

Ans:

Note: 2a=V'V=4→ a=2, $c=CF=2\sqrt{5}\Rightarrow b=\sqrt{c^2-a^2}=\sqrt{20-4}=\sqrt{16}=4$ **Ans:** $\dfrac{y^2}{4}-\dfrac{x^2}{16}=1$



Note: When centre at **C(h,k):**

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$$ This is for **horizontal orientation**

$$\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$$ This is for **vertical orientation**

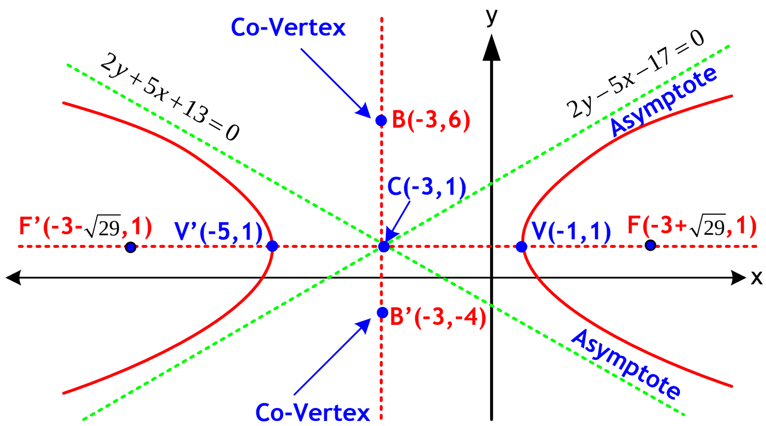
Example: Find the coordinates of the centre, vertices, and foci, the lengths of transverse and conjugate axes, the eccentricity, and the equation of the directrices and asymptotes of the hyperbola: $\frac{(x+3)^2}{4}-\frac{(y-1)^2}{25}=1$

Ans: $\frac{(x+3)^2}{4}-\frac{(y-1)^2}{25}=1$ **...(1)**

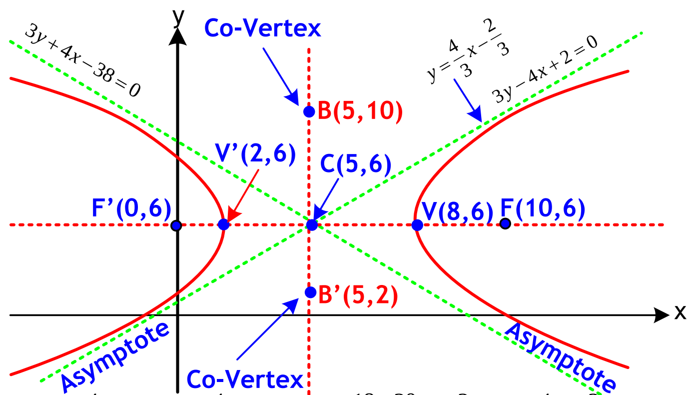
$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$ **...(2)**

Comparing **Eq.1** and **Eq.2:**

C(h,k)=C(-3,1), a=2, b=5, c= $\sqrt{4+25}=\sqrt{29}$. The following diagram explains everything in great detail.



Example: Find the equation of the hyperbola that has a centre at (5,6), a focus at (0,6) and a vertex at (8,6)



$y=\frac{4}{3}x+k \Rightarrow 6=\frac{4}{3}(5)+k \Rightarrow k=\frac{18-20}{3}=-\frac{2}{3} \Rightarrow y=\frac{4}{3}x-\frac{2}{3}$

Ans: a = CV=3, c=CF=5, then $b=\sqrt{c^2-a^2}=\sqrt{25-9}=\sqrt{16}=4$

Therefore, $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=\frac{(x-5)^2}{9}-\frac{(y-6)^2}{16}=1$

Example: For each of the following hyperbolas, find coordinates of the centre, vertices, and the foci, the length of the transverse and conjugate axes; the eccentricity; and the equations of the directrices and asymptotes. Sketch each locus.

(a) $\frac{x^2}{16}-\frac{y^2}{4}=1$ **(b)** $25y^2-9x^2=225$

(c) $9x^2-4y^2-36x+32y+8=0$ **(d)** $y^2+4y+20x+4=0???$

Ans:

(a) $\frac{x^2}{16}-\frac{y^2}{4}=1$ **...(1)**

$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ **...(2)**

Comparing **Eq.1** and **Eq.2**:

$a^2=16,\ b^2=4,$ **and** $c=\sqrt{a^2+b^2}=\sqrt{16+4}=\sqrt{20}=\sqrt{4\times5}=2\sqrt{5}$

The centre is at the **origin** and the transverse axis is along the x-axis $\left(a^2\ under\ x^2\right)$. The vertices are on the transverse axis at a distance **a = 4** from the centre; their coordinates are **V(4,0)** and **V'(-4,0).** The extremities of the conjugate axis are on the y axis at a distance b = 2 from the centre; their coordinates are **B(0,2)** and **B'(0,-2).**

The foci are on the transverse axis at a distance $c=2\sqrt{5}$ from the centre; their coordinates are $F\left(2\sqrt{5},0\right)$ and $F'\left(-2\sqrt{5},0\right)$.

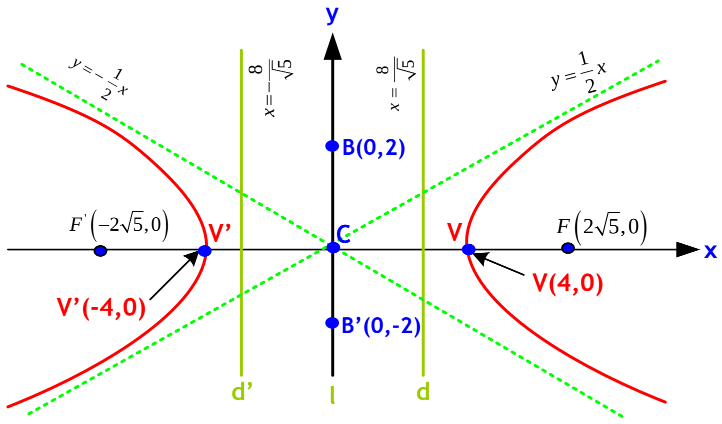
The length of the transverse and conjugate axes are $2a=8$ and $2b=4$ respectively.

The eccentricity is $e=\frac{c}{a}=\frac{2\sqrt{5}}{4}=\frac{\sqrt{5}}{2}$

The directrices are perpendicular to the transverse axis and at a distance $\frac{a^2}{c}=\frac{16}{2\sqrt{5}}=\frac{8}{\sqrt{5}}$ from the centre; their equations are:

$x=\pm\frac{8}{\sqrt{5}}$.

The equations for the asymptotes are $\frac{x^2}{16}-\frac{y^2}{4}=0 \Rightarrow y=\pm\frac{1}{2}x$



(b) $25y^2-9x^2=225$

$\frac{25y^2}{225}-\frac{9x^2}{225}=1$ **...(1)**

$\frac{y^2}{9}-\frac{x^2}{25}=1$ **...(1)**

$\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$ **...(2)**

Comparing **Eq.1** and **Eq.2**:

$a^2=9 \Rightarrow a=3,\ b^2=25 \Rightarrow b=5,$ **and** $c=\sqrt{a^2+b^2}=\sqrt{9+25}=\sqrt{34}$

The centre is at the origin and the transverse axis is along the y-axis $(a^2\ under\ y^2)$. The vertices are on the transverse axis at a distance **a= 3** from the centre; their coordinates are **V(0,3)** and **V'(0,-3).** The extremities of the conjugate axis are on the **x-axis** at a distance **b = 5** from the centre; their coordinates are **B(5,0)** and **B'(-5,0).**

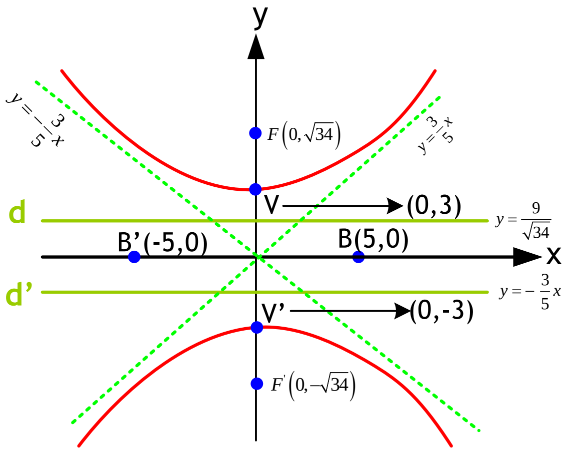
The foci are on the transverse axis at a distance $c=\sqrt{34}$ from the centre; their coordinates are $F(0,\sqrt{34})$ and $F'(0,-\sqrt{34})$.

The length of the transverse and conjugate axes are $2a=6$ and $2b=10$ respectively.

The eccentricity is $e=\frac{c}{a}=\frac{\sqrt{34}}{3}$

The directrices are perpendicular to the transverse axis and at a distance $\frac{a^2}{c}=\frac{9}{\sqrt{34}}$ from the centre; their equations are: $y=\pm\frac{9}{\sqrt{34}}$.

The equations for the asymptotes are $\frac{y^2}{9}-\frac{x^2}{25}=0 \Rightarrow y=\pm\frac{3}{5}x$



(c) $9x^2-4y^2-36x+32y+8=0$

$9x^2-36x-4y^2+32y+8=0$

$9\left(x^2-4x+4\right)-36-4\left(y^2-8y+16\right)+64+8=0$

$9(x-2)^2-4(y-4)^2+36=0$

$9(x-2)^2-4(y-4)^2=-36$

$4(y-4)^2-9(x-2)^2=36$

$\frac{4(y-4)^2}{36}-\frac{9(x-2)^2}{36}=1$

$\frac{(y-4)^2}{9}-\frac{(x-2)^2}{4}=1$ **...(1)**

$\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$ **...(2)**

Comparing **Eq.1** and **Eq.2**:

$a^2=9\Rightarrow a=3,\ \ b^2=4\Rightarrow b=2,$ **and** $c=\sqrt{a^2+b^2}=\sqrt{9+4}=\sqrt{13}$

The **centre** is at the point **C(2,4)** and the transverse axis is parallel to the y-axis along the line **x = 2**. The vertices are on the transverse axis at a distance **a= 3** from the centre; their coordinates are **V(2,7)** and **V'(2,1).** The extremities of the conjugate axis are on the line y=4 at a distance **b = 2** from the centre; their coordinates are **B(4,4)** and B'(0,4).

The foci are on the transverse axis at a distance $c=\sqrt{13}$ from the centre; their coordinates are $F\left(2,4+\sqrt{13}\right)$ and $F'\left(2,4-\sqrt{13}\right)$.

The length of the transverse and conjugate axes are $2a=6$ and $2b=4$ respectively.

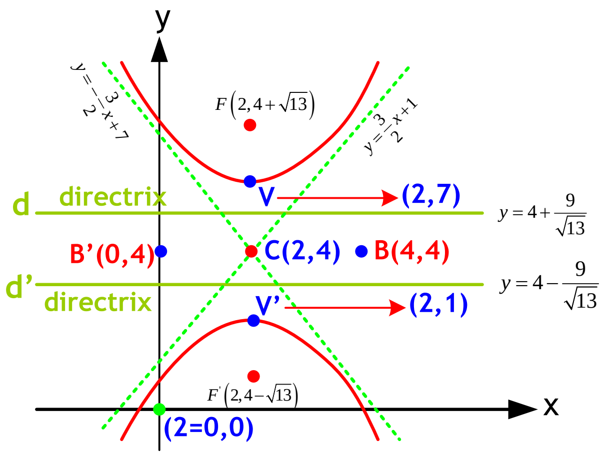
The eccentricity is $e=\frac{c}{a}=\frac{\sqrt{13}}{3}$

The directrices are perpendicular to the transverse axis and at a distance $\frac{a^2}{c}=\frac{9}{\sqrt{13}}$ from the centre; their equations are:

$$y=4\pm\frac{9}{\sqrt{13}}.$$

The equations for the asymptotes are

$$\frac{(y-4)^2}{9}-\frac{(x-2)^2}{4}=0\Rightarrow y=\frac{3}{2}x+1 \text{ or } y=-\frac{3}{2}x+7$$



Example: Find the equation of the hyperbola, given

(a) Centre C(0,0), vertex (4,0), focus (5,0)

(b) Centre C(0,0), focus (0,-4), eccentricity = 2.

(c) Centre C(0,0), vertex (5,0), one asymptote 5y+3x=0

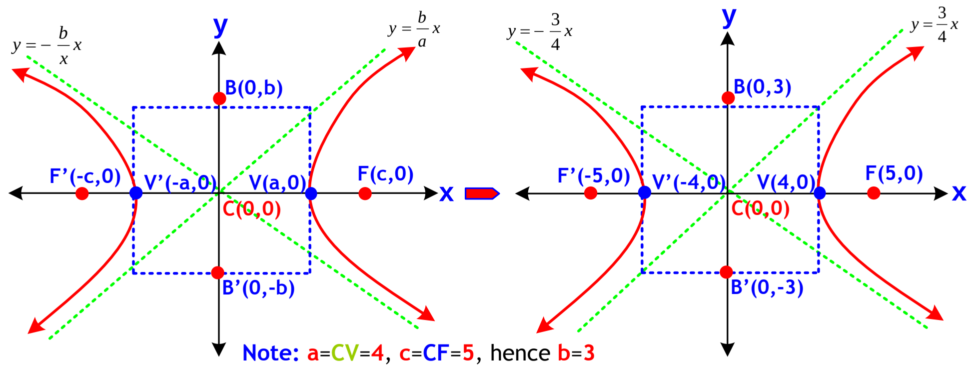
(d) Centre C(-5,4), vertex (-11,4), eccentricity = 5/3.

(e) Vertices (2,1) and (6,1), one asymptote 4y-3x+8=0

(f) Transverse axis parallel to the x-axis, asymptotes 5y-4x-8=0 and 5y+4x-32=0, passes through (5,5).

Ans:

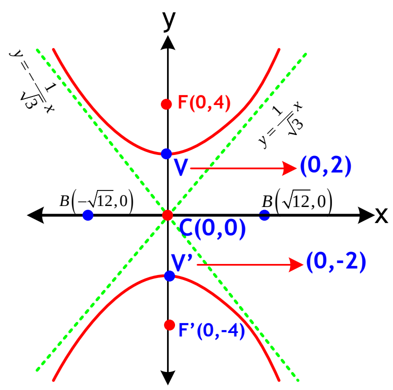
(a) **Because of Centre, vertices, and foci, it is along x-axis.**



Note: a=CV=4, c=CF=5, hence b=3

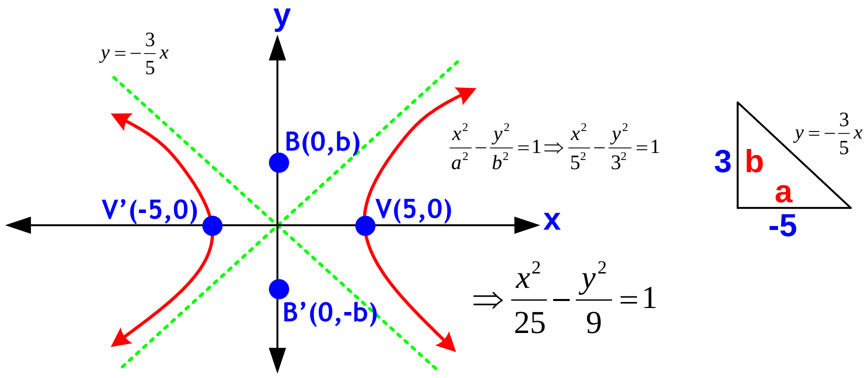
The equation of the hyperbola is: $\frac{x^2}{16}-\frac{y^2}{9}=1$

(b) Centre C(0,0), focus (0,-4), eccentricity = 2.

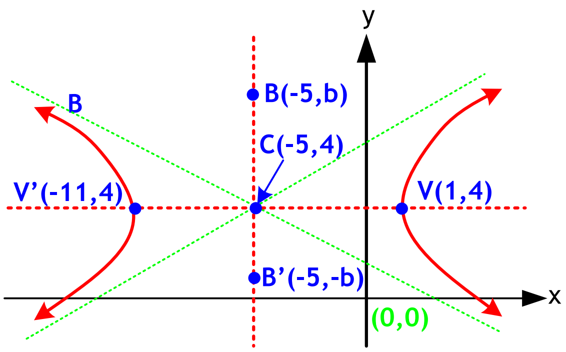


Note: c=F'C=4, e=c/a=2, --> a=2 and $b^2=c^2-c^2=12$.
The transverse axis is along the y-axis and the equation of the hyperbola is: $\frac{y^2}{4}-\frac{x^2}{12}=1$

(c) Centre C(0,0), vertex(5,0), one asymptote 5y + 3x = 0.

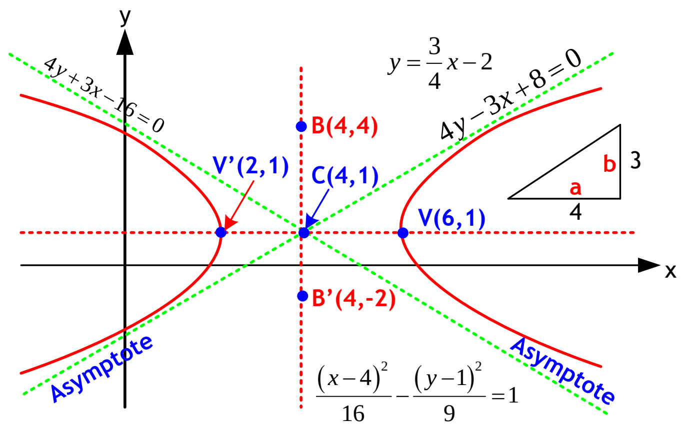


(d) Centre C(-5,4), vertex (-11,4), eccentricity = 5/3.



Note: a=V'C=6, e=c/a=5/3,
--> c=10 and b=8.
The transverse axis is along the x-axis and the equation of the hyperbola is: $\frac{(x+5)^2}{36}-\frac{(y-4)^2}{64}=1$

(e) Centre will be at the midpoint of V'V: → C(4,1) and gradient = 3/4



(f) The centre will be at the intersection of the 2 asymptotes.

$$y=\frac{4}{5}x+\frac{8}{5} \quad \text{...(1)}$$

$$y=-\frac{4}{5}x+\frac{32}{5} \quad \text{...(2)}$$

$$\frac{4}{5}x+\frac{8}{5}=-\frac{4}{5}x+\frac{32}{5}\Rightarrow\frac{8}{5}x=\frac{24}{5}\Rightarrow x=3 \text{ and}$$

$$y=\frac{4}{5}x+\frac{8}{5}=\frac{4}{5}\times3+\frac{8}{5}=\frac{20}{5}=4\Rightarrow C(3,4)$$

$$\frac{(x-3)^2}{a^2}-\frac{(y-4)^2}{b^2}=1 \quad \text{and} \quad y=mx+c=\frac{4}{5}x+\frac{8}{5}\Rightarrow m=\frac{b}{a}=\frac{4}{5}\Rightarrow b=\frac{4}{5}a$$

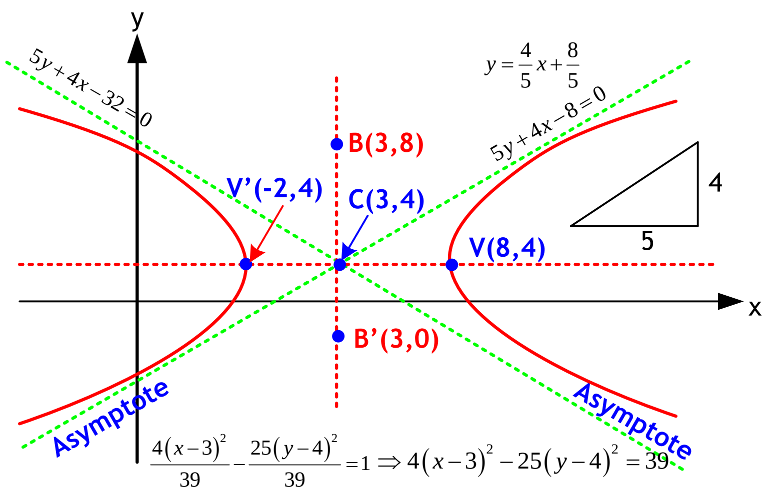
$$\frac{(x-3)^2}{a^2}-\frac{(y-4)^2}{\left(\frac{4}{5}a\right)^2}=1\Rightarrow\frac{(x-3)^2}{a^2}-\frac{(y-4)^2}{\frac{16}{25}a^2}=1$$

$$\left.\frac{(x-3)^2}{a^2}\right|_{x=5}-\left.\frac{25(y-4)^2}{16a^2}\right|_{y=5}=1$$

$$\frac{(5-3)^2}{a^2}-\frac{25(5-4)^2}{16a^2}=1\Rightarrow\frac{4}{a^2}-\frac{25}{16a^2}=1\Rightarrow\frac{39}{16a^2}=1\Rightarrow16a^2=39\Rightarrow a=\frac{\sqrt{39}}{4}$$

$$b=\frac{4}{5}a=\frac{4}{5}\left(\frac{\sqrt{39}}{4}\right)=\frac{\sqrt{39}}{5}\Rightarrow\frac{4(x-3)^2}{39}-\frac{25(y-4)^2}{39}=1$$

$$\Rightarrow4(x-3)^2-25(y-4)^2=39$$



Exercises

1. For each of the following parabolas, sketch the curve, find coordinates of the vertex and focus, and find the equation of the axis and directrix.

   (a) $y^2 = -8x$ **(b)** $x^2 = 8y$

   (c) $x^2 - 4y - 6x - 3 = 0$ **(d)** $y^2 - 8x - 2y + 17 = 0$

2. Find the equation of the parabola, given

   (a) V(0,0); F(0,2) **(b)** V(0,0); directrix: x=5

   (c) V(-2,4); F(1,4) **(d)** F(2,2); directrix: y=6

   (e) V(1,1); axis: x=1; passing through (3,-1)

3. For each of the following hyperbolas, find coordinates of the centre, vertices, and the foci, the length of the transverse and conjugate axes; the eccentricity; and the equations of the directrices and asymptotes. Sketch each locus.

   (a) $\frac{x^2}{9} - \frac{y^2}{4} = 1$ **(b)** $16y^2 - 9x^2 = 144$

   (c) $9y^2 - 4x^2 - 72y + 16x - 36 = 0$

4. Find the equation of the hyperbola, given

   (a) Centre C(0,0), focus (0,-3), eccentricity = 1.5.

   (b) Centre C(0,0), vertex (3,0), focus (4,0)

   (c) Centre C(-5,3), vertex (-9,3), eccentricity = 5/4.

   (d) Centre C(0,0), vertex(4,0), one asymptote 4y - 3x = 0.

   (e) Transverse axis parallel to the x-axis, asymptotes 3x+y-7=0 and 3x-y-5=0, passes through (4,4).

   (f) Vertices (4,2) and (10,2), one asymptote 3y-4x+22=0

5. Find the equation of the hyperbola that has a centre at (8,2), a focus at (3,2) and a vertex at (11,2)

6. Write the equation of the conjugate of the hyperbola $16y^2 - 25x^2 = 400$

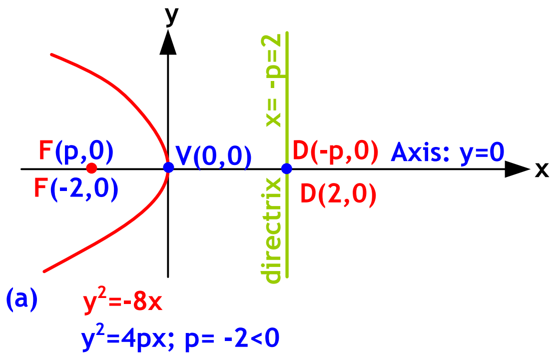
Exercises and Solutions

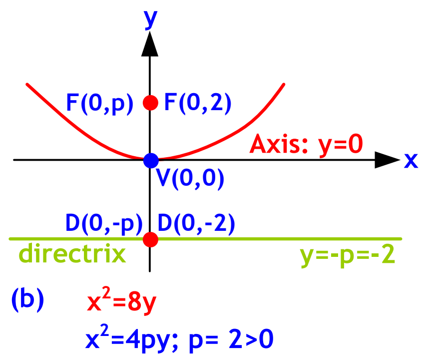
1. For each of the following parabolas, sketch the curve, find coordinates of the vertex, focus, and find the equation of the axis and directrix.

(a) $y^2 = -8x$ **(b)** $x^2 = 8y$

(c) $x^2 - 4y - 6x - 3 = 0$ **(d)** $y^2 - 8x - 2y + 17 = 0$

Ans:





(c) $x^2 - 4y - 6x - 3 = 0$

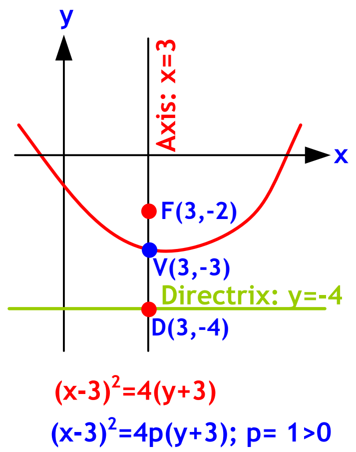
$x^2 - 6x + 9 - 9 - 4y - 3 = 0$

$x^2 - 6x + 9 - 4y - 12 = 0$

$(x-3)^2 = 4(y+3)$ **...(1)**

$(x-h)^2 = 4p(y-k)$ **...(2)**

Comparing **Eq.1** and **Eq.2**: → V(h,k)=V(3,-3); 4p=4 → **p =1>0**



(d) $y^2-8x-2y+17=0$

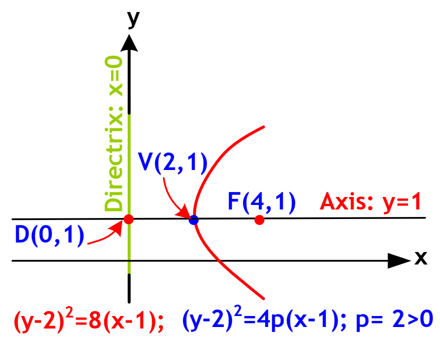
$y^2-2y+1-1-8x+17=0$

$y^2-2y+1-8x+16=0$

$(y-1)^2=8(x-2)$ ...(1)

$(y-k)^2=4p(x-h)$ ...(2)

Comparing **Eq.1** and **Eq.2**: → V(h,k)=V(2,1); 4p=8 → **p =2>0**



2. Find the equation of the parabola, given:

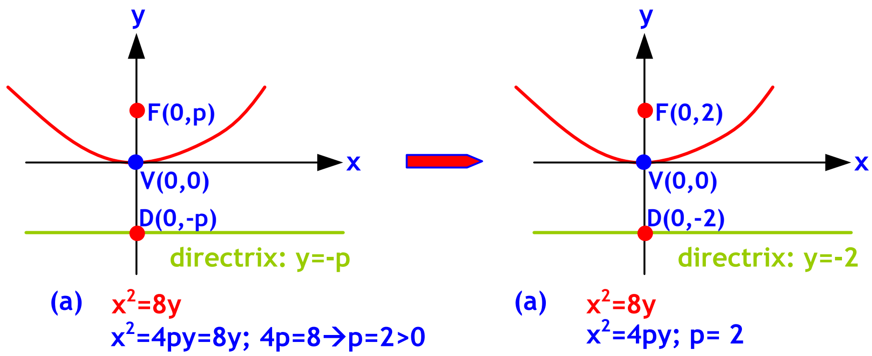
(a) V(0,0); F(0,2) **(b)** V(0,0); directrix: x=5

(c) V(-2,4); F(1,4) **(d)** F(2,2); directrix: y=6

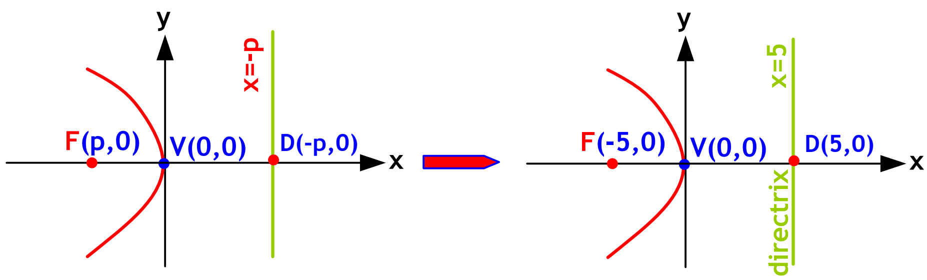
(e) V(1,1); axis: x=1; passing through (3,-1)

Ans:

(a) **Note:** Since the directed distance **p=VF=2,** the parabola opens upward, and the equation is: $x^2=4py=8y$

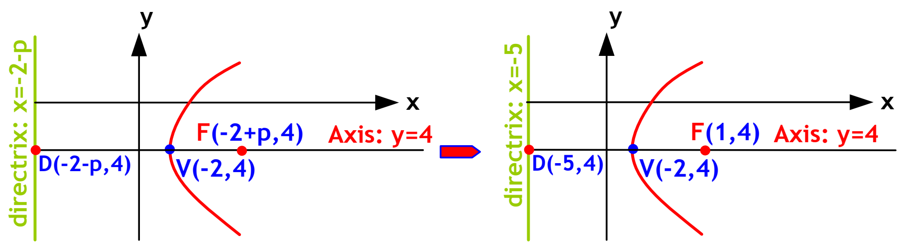


(b) **Note:** Here the directrix lies to the right of the vertex, so the parabola opens to the left. The directed distance **$p=DV=-5$** and the equation is: **$y^2=4px=-20x$.**



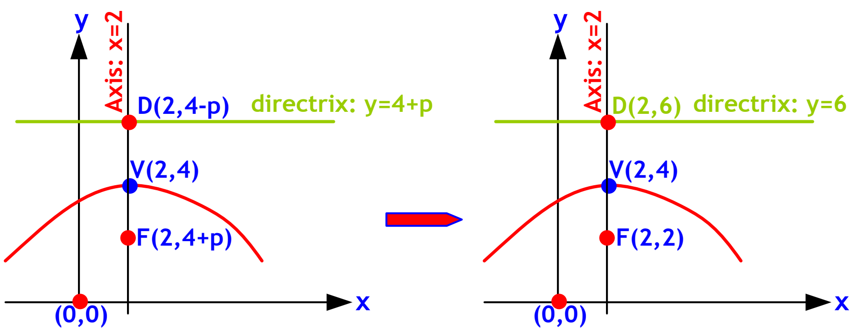
(b) $y^2=-20x=4px;\ p= -5<0$ (b) $y^2=-20x=4px;\ p= -5<0$

(c) **Note:** Here the focus lies to the right of the vertex, and the parabola opens to the right. The directed distance **$p=VF=3$** and the equation is: **$(y-4)^2=12(x-1)$.**



(c) $(y-4)^2=4p(x+2)=12(x+2);\ p=3>0$ (c) $(y-4)^2=4p(x+2)=12(x+2);\ p=3>0$

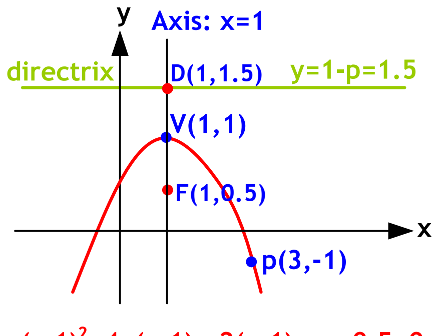
(d) **Note:** Here the focus lies below the directrix and the parabola opens downward. The axis of the parabola meets the directrix in D(2,6) and the vertex is at the midpoint V(2,4) of FD. Then p=VF=-2 and the equation is: **$(x-2)^2=-8(y-4)$.**



(d) $(x-2)^2=4p(y-4);\ F(2,4-p)=F(2,2)$
$\rightarrow 4+p=2 \rightarrow p=2-4=-2$

(d) $(x-2)^2=-8(y-4)$

(e) **Note:** The equation of this parabola is of the form **(x-1)^2=4p(y-1).** If (3,-1) is a point on the parabola, then **(3-1)^2=4p(-1-1), →-8p=4 →p=-0.5,** and the equation is: **(x-1)^2=4p(y-1)=-2(y-1).**



3. For each of the following hyperbolas, find coordinates of the centre, vertices, and the foci, the length of the transverse and conjugate axes; the eccentricity; and the equations of the directrices and asymptotes. Sketch each locus.

(a) $\frac{x^2}{9}-\frac{y^2}{4}=1$ **(b)** $16y^2-9x^2=144$

(c) $9y^2-4x^2-72y+16x-36=0$

Ans:

(a) $\frac{x^2}{9}-\frac{y^2}{4}=1$ **...(1)**

$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ **...(2)**

Comparing **Eq.1** and **Eq.2**:

$a^2=9\Rightarrow a=3,\ \ b^2=4\Rightarrow b=2,$ **and** $c=\sqrt{a^2+b^2}=\sqrt{9+4}=\sqrt{13}$

The centre is at the origin and the transverse axis is along the x-axis $\left(a^2\ under\ x^2\right)$. The vertices are on the transverse axis at a distance **a = 3** from the centre; their coordinates are V(3,0) and V'(-3,0). The extremities of the conjugate axis are on the **y-axis** at a distance b = 2 from the centre; their coordinates are B(0,2) and B'(0,-2).

The foci are on the transverse axis at a distance $c=\sqrt{13}$ from the centre; their coordinates are $F\left(\sqrt{13},0\right)$ and $F'\left(-\sqrt{13},0\right)$.

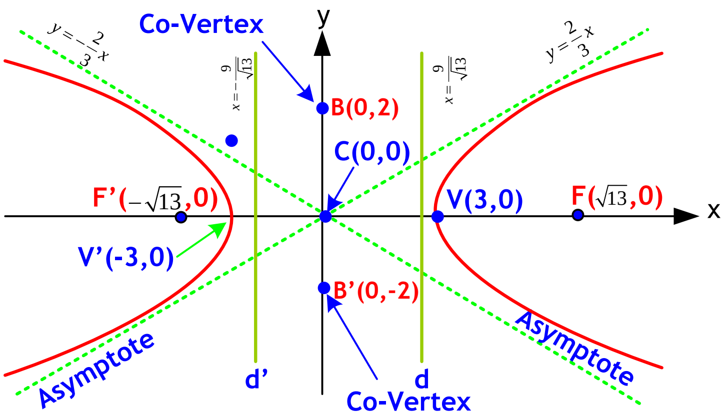
The length of the transverse and conjugate axes are $2a = 6$ and $2b = 4$ respectively.

The eccentricity is $e = \frac{c}{a} = \frac{\sqrt{13}}{3} = \frac{\sqrt{13}}{3}$

The directrices are perpendicular to the transverse axis and at a distance $\frac{a^2}{c} = \frac{9}{\sqrt{13}} = \frac{9}{\sqrt{13}}$ from the centre; their equations are:

$x = \pm\frac{9}{\sqrt{13}}$.

The equations for the asymptotes are $\frac{x^2}{9} - \frac{y^2}{4} = 0 \Rightarrow y = \pm\frac{2}{3}x$



(b) $16y^2 - 9x^2 = 144$

$$\frac{16y^2}{144} - \frac{9x^2}{144} = 1 \qquad \text{...(1)}$$

$$\frac{y^2}{9} - \frac{x^2}{16} = 1 \qquad \text{...(1)}$$

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1 \qquad \text{...(2)}$$

Comparing **Eq.1** and **Eq.2**:

$a^2 = 9 \Rightarrow a = 3,\ \ b^2 = 16 \Rightarrow b = 4,$ **and** $c = \sqrt{a^2 + b^2} = \sqrt{9+16} = \sqrt{25} = 5$

The centre is at the origin and the transverse axis is along the y-axis $\left(a^2\ under\ y^2\right)$. The vertices are on the transverse axis at a distance **a= 3** from the centre; their coordinates are **V(0,3)** and **V'(0,-3).** The extremities of the conjugate axis are on the **x-axis** at a distance **b = 4** from the centre; their coordinates are **B(4,0)** and **B'(-4,0).**

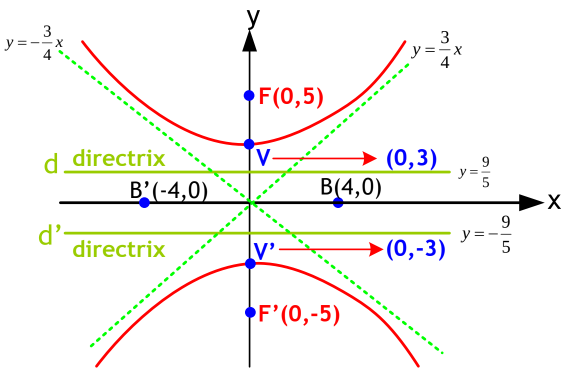
The foci are on the transverse axis at a distance **c=5** from the centre; their coordinates are **F(0,5)** and **F(0,-5)**.

The length of the transverse and conjugate axes are **2a=6** and **2b=8** respectively.

The **eccentricity** is $e=\frac{c}{a}=\frac{5}{3}$

The directrices are perpendicular to the transverse axis and at a distance $\frac{a^2}{c}=\frac{9}{5}$ from the centre; their equations are: $y=\pm\frac{9}{5}$.

The equations for the asymptotes are $\frac{y^2}{9}-\frac{x^2}{16}=0 \Rightarrow y=\pm\frac{3}{4}x$



(c) $9y^2-4x^2-72y+16x-36=0$

$9y^2-72y-4x^2+16x-36=0$

$9\left(y^2-8y\right)-4\left(x^2-4x\right)-36=0$

$9\left(y^2-8y+16-16\right)-4\left(x^2-4x+4-4\right)-36=0$

$9\left(y^2-8y+16\right)-144-4\left(x^2-4x+4\right)+16-36=0$

$9\left(y-4\right)^2-4\left(x-2\right)^2-36=0$

$$9(y-4)^2-4(x-2)^2=36$$

$$\frac{9(y-4)^2}{36}-\frac{4(x-2)^2}{36}=1$$

$$\frac{(y-4)^2}{4}-\frac{(x-1)^2}{9}=1 \qquad \text{...(1)}$$

$$\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1 \qquad \text{...(2)}$$

Comparing **Eq.1** and **Eq.2**:

$$a^2=4\Rightarrow a=2,\ \ b^2=9\Rightarrow b=3,\ \text{and}\ c=\sqrt{a^2+b^2}=\sqrt{4+9}=\sqrt{13}$$

The centre is at the point **C(1,4)** and the transverse axis is parallel to the y-axis along the line **x = 1**. The vertices are on the transverse axis at a distance **a= 2** from the centre; their coordinates are **V(1,6)** and **V'(1,2).** The extremities of the conjugate axis are on the line **y=4** at a distance **b = 3** from the centre; their coordinates are **B(4,4)** and **B'(0,4).**

The foci are on the transverse axis at a distance $c=\sqrt{13}$ from the centre; their coordinates are $F\left(2,4+\sqrt{13}\right)$ and $F'\left(2,4-\sqrt{13}\right)$.

The length of the transverse and conjugate axes are $2a=6$ and $2b=4$ respectively.

The eccentricity is $e=\frac{c}{a}=\frac{\sqrt{13}}{3}$

The directrices are perpendicular to the transverse axis and at a distance $\frac{a^2}{c}=\frac{9}{\sqrt{13}}$ from the centre; their equations are:

$$y=4\pm\frac{9}{\sqrt{13}}.$$

The equations for the asymptotes are

$$\frac{(y-4)^2}{9}-\frac{(x-2)^2}{4}=0\Rightarrow y=\frac{3}{2}x+1\ \text{ or }\ y=-\frac{3}{2}x+7$$

4. Find the equation of the hyperbola, given

(a) Centre C(0,0), focus (0,-3), eccentricity = 1.5.

(b) Centre C(0,0), vertex (3,0), focus (4,0)

(c) Centre C(-5,3), vertex (-9,3), eccentricity = 5/4.

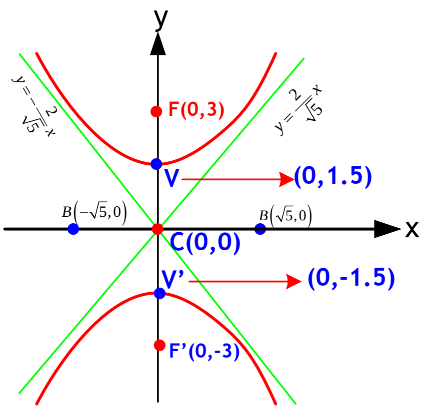
(d) Centre C(0,0), vertex(4,0), one asymptote 4y - 3x = 0.

(e) Transverse axis parallel to the x-axis, asymptotes 3x+y-7=0 and 3x-y-5=0, passes through (4,4).

(f) Vertices (4,2) and (10,2), one asymptote 3y-4x+22=0

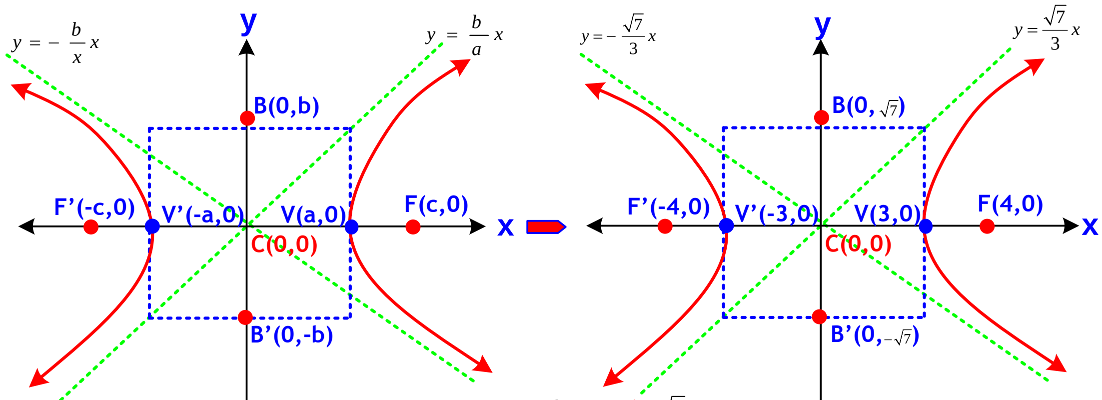
Ans:

(a) Centre C(0,0), focus (0,-3), eccentricity = 1.5.



Note: c=F'C=3, e=c/a=1.5, --> a=2 and $b^2=c^2-c^2=5$.
The transverse axis is along the y-axis and the equation of the hyperbola is: $\frac{y^2}{4}-\frac{x^2}{5}=1$

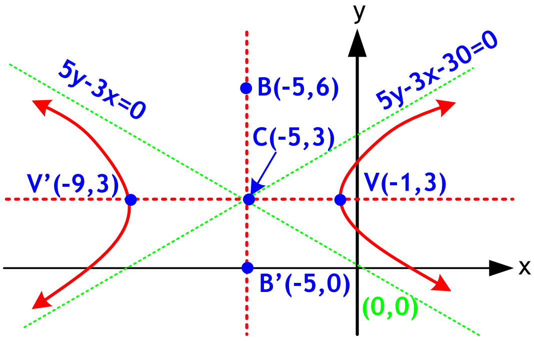
(b)



Note: a=CV=3, c=CF=4, hence $b=\sqrt{7}$

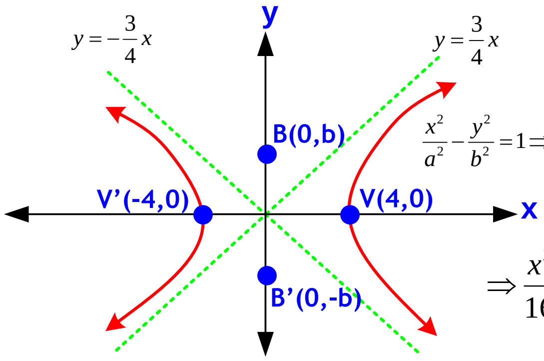
The equation of the hyperbola is: $\frac{x^2}{9}-\frac{y^2}{7}=1$

(c)

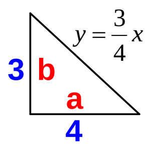


Note: a=V'C=4, e=c/a=5/4,
--> c=5 and b=3.
The transverse axis is along the x-axis and the equation of the hyperbola is: $\frac{(x+5)^2}{25}-\frac{(y-3)^2}{9}=1$

(d)



$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1\Rightarrow\frac{x^2}{4^2}-\frac{y^2}{3^2}=1$



$\Rightarrow\frac{x^2}{16}-\frac{y^2}{9}=1$

(e) The centre will be at the intersection of the 2 asymptotes.

$y=-3x+7$ ...(1)

$y=3x-5$ ...(2)

$-3x+7=3x-5\Rightarrow 6x=12\Rightarrow x=2$ and $y=3x-5=3\times2-5=1\Rightarrow C(2,1)$

$\frac{(x-2)^2}{a^2}-\frac{(y-1)^2}{b^2}=1$ and $y=mx+c=3x-5\Rightarrow m=\frac{b}{a}=3\Rightarrow b=3a$

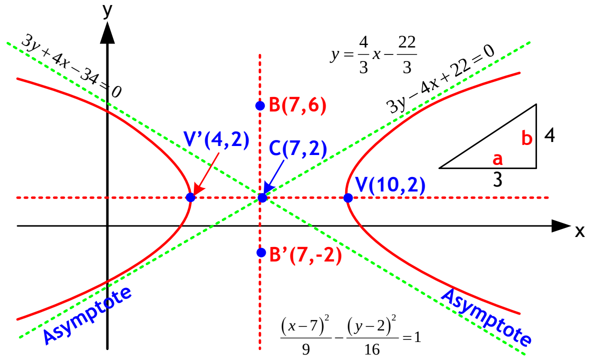
$$\frac{(x-2)^2}{a^2}-\frac{(y-1)^2}{(3a)^2}=1\Rightarrow\frac{(x-2)^2}{a^2}-\frac{(y-1)^2}{9a^2}=1$$

$$\frac{(x-2)^2}{a^2}\bigg|_{x=4}-\frac{(y-1)^2}{9a^2}\bigg|_{y=4}=1$$

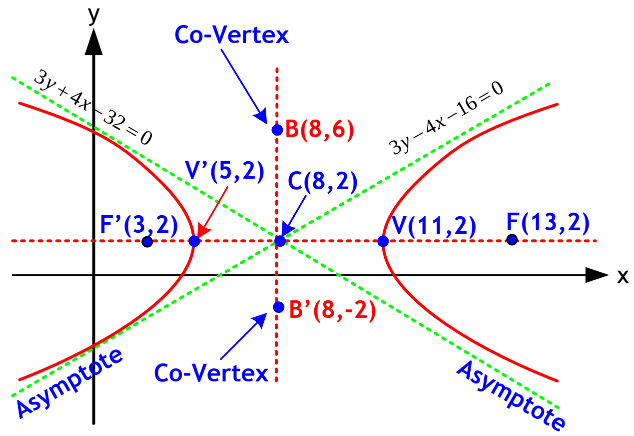
$$\frac{(4-2)^2}{a^2}-\frac{(4-1)^2}{9a^2}=1\Rightarrow\frac{4}{a^2}-\frac{9}{9a^2}=1\Rightarrow\frac{4}{a^2}-\frac{1}{a^2}=\frac{3}{a^2}=1\Rightarrow a=\sqrt{3}$$

$$b=3a=3\sqrt{3}\Rightarrow\frac{(x-2)^2}{3}-\frac{(y-1)^2}{27}=1$$

(f) Centre will be at the midpoint of V'V: → C(7,2) and gradient = 4/3



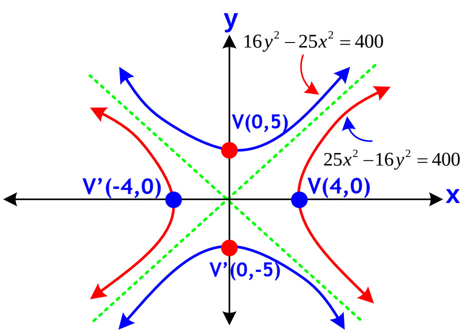
5. Find the equation of the hyperbola that has a centre at (8,2), a focus at (3,2) and a vertex at (11,2)



Ans: a=CV=3, c=CF=5, then b =4 $\frac{(x-8)^2}{9}-\frac{(y-2)^2}{16}=1$

6. Write the equation of the conjugate of the hyperbola $16y^2-25x^2=400$

Note: The equation of the conjugate hyperbola is: $25x^2-16y^2=400$. The common asymptotes have equations $y=\pm\frac{5}{4}x$. The vertices of $16y^2-25x^2=400$ are at $(0,\pm 5)$. The vertices of $25x^2-16y^2=400$ are: at $(\pm 4,0)$. The curve is shown below.



الله(ج) د وکړي چې تاسي ما ته، زما مور او پلار ته، زما اولادونو ته، زما ټولي کورنۍ اوټولو مسلمانانو ته د زړه له کومي دعا کړيې وي، **اوکنه؟** نو اوس يې لطفاً وکړئ!

په پای کي له تاسو ټولو سره له لرې ځایه خدای په اماني کوم. له الله(ج) څخه هيله کوم چې په افغانستان او په ټوله نړۍ کي هغه شانته صلح راشي، چې الله(ج) ته مقبوله او منظوره وي، او الله(ج) دې، د افغانانو ترمنځ د حق فيصله وکړي. آمين يارب الٰعلمين.